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Inversion identities for inhomogeneous face models

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Abstract

We derive exact inversion identities satisfied by the transfer matrix of inhomogeneous interaction-round-a-face (IRF) models with arbitrary boundary conditions using the underlying integrable structure and crossing properties of the local Boltzmann weights. For the critical restricted solid-on-solid (RSOS) models these identities together with some information on the analytical properties of the transfer matrix determine the spectrum completely and allow to derive the Bethe equations for both periodic and general open boundary conditions.

1. Introduction

Functional relations between the transfer matrices of integrable models together with the knowledge of their analytical properties provide a powerful basis for the solution of their spectral problem. An important example are the so-called inversion relations [1–3]. In the thermodynamic limit these relations become identities (at least for part of the spectrum) allowing to compute the free energy of certain models exactly [4,5]. Generalized inversion relations for the restricted solid-on-solid (RSOS) model have been obtained from the fusion hierarchy [6,7] and have been used to identify the low energy effective theory of the critical model through solution of nonlinear integral equations [8] or to study their surface critical behavior [9].

Recently, sets of *exact* inversion identities for the transfer matrices of inhomogeneous vertex models have been used to tackle the long-standing problem of finding Bethe equations for the

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spectrum of the integrable XXZ spin chain subject to non-diagonal boundary conditions which break the $U(1)$ symmetry of the bulk, see e.g. Refs. [10–19]. They have been derived for vertex models using as only input the underlying Yang–Baxter and reflection equation and physical assumptions such as crossing and unitary of the local Boltzmann weights. Unlike the inversion identities mentioned above they only hold for a discrete set of spectral parameters related to the inhomogeneities introduced in the lattice model [20]. Similar expressions for the corresponding eigenvalues had been obtained before using Sklyanin's separation of variables [16,21], or by considering certain matrix elements of the transfer matrix [22,23].

While the use of these identities for the actual computation of eigenvalues is restricted to small systems they allow, once complemented by information on the analytical properties of the transfer matrix, to formulate the spectral problem in the form of Baxter's TQ-equation [24] or inhomogeneous generalizations thereof [20,22,23]. In addition, the number of solutions to the inversion identities are rather easily counted which allows to address the problem of completeness of the Bethe ansatz for the underlying model [16,21,25–27].

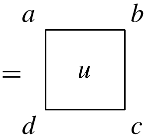
First attempts to extend this method for the solution of the spectral problem to integrable interaction-round-a-face (IRF) statistical models have made use of the reorganization of Boltzmann weights of solid-on-solid (SOS) models in an R-matrix solving the dynamical six-vertex Yang-Baxter algebra [28]. Adapting Sklyanin's separation of variables the eigenvalues of the transfer matrix of the dynamical six-vertex model on a lattice with odd number of sites and with antiperiodically twisted boundary conditions have been shown to satisfy quadratic equations for a discrete set of spectral parameters [29]. In another approach functional relations have been derived from the dynamical Yang–Baxter equation to determine the partition function of the SOS model with domain wall boundaries [30,31].

In this paper we derive inversion identities for the transfer matrix of general IRF models directly in the face formulation of the Yang–Baxter algebra using unitarity and crossing properties of the local Boltzmann weights. For this we consider inhomogeneous face models subject to periodic and generic integrable open boundary conditions. Starting from these identities we show that they allow to derive TQ-equations for the critical RSOS models. The eigenvalues of the transfer matrix are parametrized in terms of the solution to Bethe equations which allow to study properties of finite chains and to perform the thermodynamic limit.

2. Discrete inversion identities for IRF models

Below we consider inhomogeneous IRF models and construct inversion identities satisfied by their commuting transfer matrices. As will become transparent below, the derivation is valid for a generic class of integrable lattice models, provided that certain local relations are satisfied. The fundamental blocks of the models are given by the Boltzmann face weights

$$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, u\right) =$$



where the spin variables a, b, c, d take values within a discrete set $\mathfrak{S}$. The allowed states of the IRF model are constrained by selection rules which are conveniently encoded in the so-called adjacency matrix A:

$$A_{ab} = \begin{cases} 0: & \text{spins } a \text{ and } b \text{ may not be adjacent} \\ 1: & \text{spins } a \text{ and } b \text{ may be adjacent,} \end{cases}$$

such that the Boltzmann weights satisfy

$$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) = A_{ab}A_{bc}A_{cd}A_{da}\,W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right). \tag{2.1}$$

The face weights are assumed to satisfy a set of local relations. First of all, the integrability of the models is guaranteed by the Yang–Baxter equation (YBE)

$$\begin{aligned}&\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right)\\ &\quad = \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right).\end{aligned} \tag{2.2}$$

In addition, we assume that the Boltzmann weights satisfy unitarity

$$\sum_e W\left(\begin{matrix} d & e \\ a & b \end{matrix}\middle| u\right) W\left(\begin{matrix} d & c \\ e & b \end{matrix}\middle| -u\right) = \rho(u)\rho(-u)\delta_{ac}, \tag{2.3}$$

crossing symmetry

$$W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| \lambda-u\right) = W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right), \tag{2.4}$$

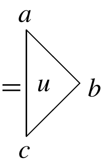
and become diagonal at the so-called shift points

$$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| 0\right) = \delta_d^b, \quad \text{and} \quad W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| \lambda\right) = \delta_c^a, \tag{2.5}$$

which correspond to the identification of scattering particles in the underlying physical picture. The function $\rho(u)$ appearing in (2.3) is model-dependent. It can be normalized such that $\rho(0) = 1$.

For models with open boundary conditions (left and right) boundary Boltzmann weights have to be introduced

$$B\left(\begin{matrix} a & \\ & b \\ c & \end{matrix}\middle| u\right) =$$



Integrability requires that they satisfy the reflection or boundary Yang–Baxter equation (BYBE). For the left boundary weights the BYBE is given by [7,32]

$$\begin{aligned}&\sum_{f,g} W\left(\begin{matrix} c & b \\ f & a \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & c \\ g & f \end{matrix}\middle| \lambda-u-v\right) B\left(\begin{matrix} g & \\ & f \\ a & \end{matrix}\middle| u\right) B\left(\begin{matrix} e & \\ & d \\ g & \end{matrix}\middle| v\right)\\ &\quad = \sum_{f,g} W\left(\begin{matrix} e & d \\ f & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} f & c \\ g & b \end{matrix}\middle| \lambda-u-v\right) B\left(\begin{matrix} e & \\ & f \\ g & \end{matrix}\middle| u\right) B\left(\begin{matrix} g & \\ & b \\ a & \end{matrix}\middle| v\right).\end{aligned} \tag{2.6}$$

The boundary weights are normalized by the boundary inversion condition

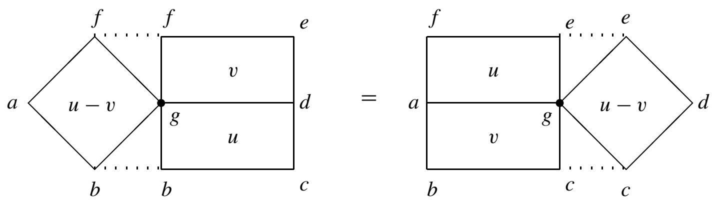
$$\sum_c B\left(\begin{matrix} a & \\ & b \\ c & \end{matrix}\middle| u\right) B\left(\begin{matrix} c & \\ & b \\ d & \end{matrix}\middle| -u\right) = \beta_a(u)\beta_a(-u)\delta_d^a, \tag{2.7}$$

with model-dependent functions $\beta_a(u)$. Furthermore, they are required to satisfy the boundary crossing condition

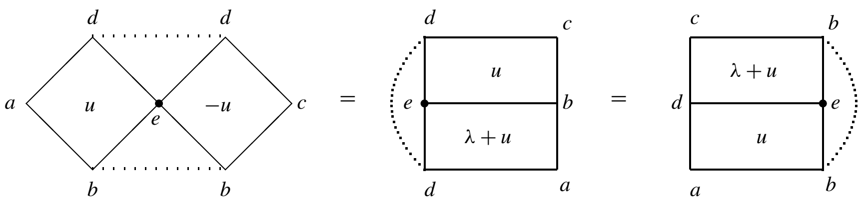
$$\sum_d B\left(\begin{matrix} c \\ a \end{matrix} d \middle| u \right) W\left(\begin{matrix} c & b \\ d & a \end{matrix} \middle| 2u-\lambda \right) = -\rho(\lambda-2u) B\left(\begin{matrix} c \\ a \end{matrix} b \middle| \lambda-u \right). \tag{2.8}$$

Similar relations hold for the right boundary weights [7].

For the derivation of inversion identities below we use a graphical representation of the relations listed above, see also [7]: the YBE (2.2) is given by



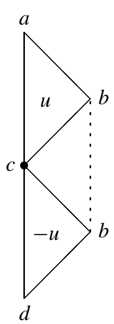
Here and in the following diagrams the spin variables on nodes with a solid circle are summed over all elements from $\mathfrak{S}$. Nodes with equal spins are connected by a dotted line. Similarly, the unitarity condition is represented by



$= \rho(u)\rho(-u)\delta_{ac}$

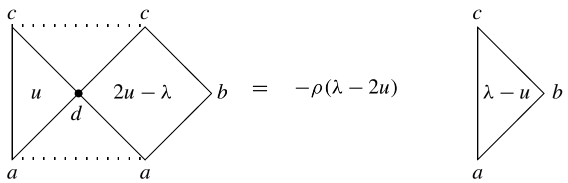
where the first diagram depicts (2.3). Crossing symmetry (2.4) has been used for the alternative representations.

Boundary inversion (2.7) and crossing condition (2.8) for the left boundary weights are represented in a similar manner by



$= \quad \delta^a_d \beta_a(u)\beta_a(-u)$

and

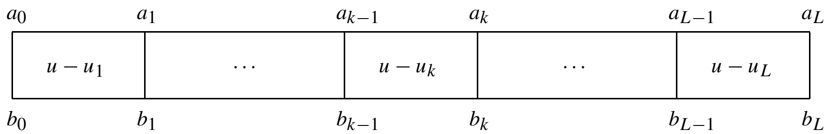


respectively. The corresponding relations for the right boundaries are obtained by reflecting these diagrams.

2.1. *Periodic boundary conditions*

To derive a set of inversion identities for integrable IRF models subject to periodic boundary conditions we introduce columns of inhomogeneities $\{u_\ell\}$. The resulting transfer matrix is given by the product of Boltzmann weights

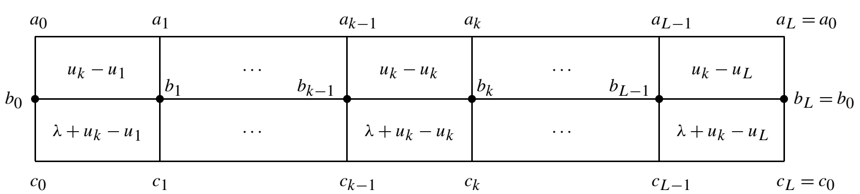
$$\mathbf{T}(u) \equiv T_{b_0\cdots b_L}^{a_0\cdots a_L}(u) = \prod_{\ell=1}^{L} W\left(\begin{matrix} a_{\ell-1} & a_\ell \\ b_{\ell-1} & b_\ell \end{matrix} \middle| u - u_\ell\right)$$

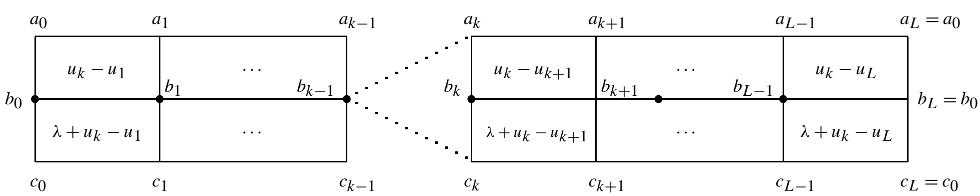
$=$ 

(2.9)

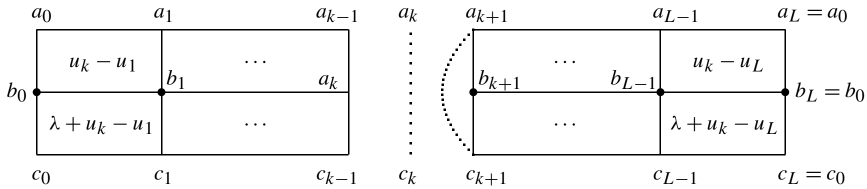
where $(a_L, b_L) \equiv (a_0, b_0)$ to impose periodic boundary conditions. As a direct consequence of the YBE (2.2) the transfer matrices form a commuting family of operators, $[\mathbf{T}(u), \mathbf{T}(v)] = 0$, which establishes the integrability of the inhomogeneous model. Using the local relations (2.3)–(2.5) then, it can be shown that the product $\mathbf{T}(u)\mathbf{T}(\lambda + u)$ becomes diagonal for $u = u_k$. Using the graphical representation introduced above, we obtain

$$T_{b_0\cdots b_L}^{a_0\cdots a_L}(u_k)T_{c_0\cdots c_L}^{b_0\cdots b_L}(\lambda + u_k)$$

$=$ 

$=$ 

$$= \rho(u_k - u_{k+1})\rho(u_{k+1} - u_k)$$

$\times$ 

$$= \cdots = \left(\prod_{\ell=1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k) \right) \delta_{c_1}^{a_1} \cdots \delta_{c_L}^{a_L}.$$

Here, the first graph represents the product of the two transfer matrices at the particular values. As a consequence of the shift points (2.5) the k-th column reduces to $\delta_{b_{k-1}}^{a_k} \delta_{c_k}^{b_{k-1}} = \delta_{b_{k-1}}^{a_k} \delta_{c_k}^{a_k}$. Using the unitarity (2.3) of the Boltzmann weights in the $(k+1)$-st column an additional Kronecker delta, $\delta_{c_{k+1}}^{a_{k+1}}$, is produced. Repetitive use of unitarity proves that the product of the transfer matrices is essentially proportional to the identity operator

$$\mathbf{T}(u_k)\mathbf{T}(\lambda + u_k) = \left(\prod_{\ell=1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k) \right) \mathbf{1}, \quad k = 1, 2, \ldots, L. \tag{2.10}$$

The number of these inversion identities is equal to the length L of the model. However, due to the identity

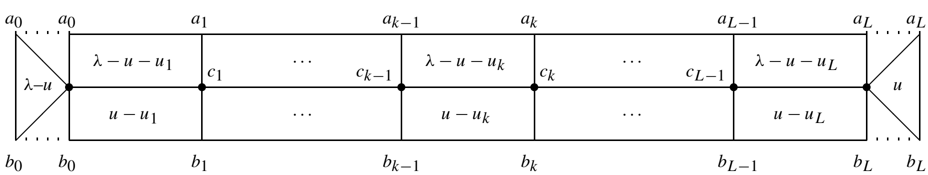
$$\prod_{\ell=1}^{L} \mathbf{T}(u_\ell) = \prod_{\ell=1}^{L} \mathbf{T}(\lambda + u_\ell) = \left(\prod_{k,\ell=1}^{L} \rho(u_k - u_\ell) \right) \mathbf{1}, \tag{2.11}$$

only $L-1$ of Eqs. (2.10) are independent. It should be noted that the homogeneous limit of the inversion identities (2.10) coincides with the $u=0$ limit of the inversion identities derived in [2] for homogeneous face models.

2.2. *Generic integrable open boundary conditions*

The commuting double-row transfer matrix for an open boundary IRF model with inhomogeneities $\{u_\ell\}$ is given by [7,33]

$$D_{b_0 \cdots b_L}^{a_0 \cdots a_L}(u) = \sum_{c_i} B\left(\begin{matrix} a_0 \\ b_0 \end{matrix} c_0 \middle| \lambda - u \right) \\ \times \left[\prod_{\ell=1}^{L} W\left(\begin{matrix} a_{\ell-1} & a_\ell \\ c_{\ell-1} & c_\ell \end{matrix} \middle| \lambda - u - u_\ell \right) W\left(\begin{matrix} c_{\ell-1} & c_\ell \\ b_{\ell-1} & b_\ell \end{matrix} \middle| u - u_\ell \right) \right] B\left(c_L \begin{matrix} a_L \\ b_L \end{matrix} \middle| u \right)$$

$=$ 

(2.12)

By construction $\mathbf{D}(u)$ enjoys crossing symmetry [7]

$$\mathbf{D}(u) = \mathbf{D}(\lambda - u). \tag{2.13}$$

The derivation of inversion identities for the case of open boundary conditions is more subtle than for the periodic chain, since apart from the crossing symmetry and unitarity relations, one has to use the YBE (2.2) and the local relations satisfied by the boundary weights, in particular boundary inversion (2.7) and crossing (2.8). Crossing symmetry (2.13) of the transfer matrix implies $\mathbf{D}(u)\mathbf{D}(\lambda + u) = \mathbf{D}(u)\mathbf{D}(-u)$. Considering the second form at $u = u_k$, for $k = 1, \ldots, L$, we find that the inversion identities satisfied by the double-row transfer matrices of IRF models are given by

$$\begin{aligned}
&D^{a_0 \cdots a_L}_{b_0 \cdots b_L}(u_k) D^{b_0 \cdots b_L}_{d_0 \cdots d_L}(-u_k) \\
&\quad = \beta_{a_L}(u_k)\beta_{a_L}(-u_k)\beta_{a_0}(u_k)\beta_{a_0}(-u_k)\rho(2u_k - \lambda)\rho(-\lambda - 2u_k) \\
&\qquad \times \prod_{\substack{\ell=1 \\ \ell \neq k}}^{L} \rho(u_k - u_\ell)\rho(-u_k + u_\ell)\rho(u_k + u_\ell)\rho(-u_k - u_\ell) \\
&\qquad \times \delta^{a_0}_{d_0}\delta^{a_1}_{d_1} \cdots \delta^{a_L}_{d_L}.
\end{aligned} \tag{2.14}$$

Note that while the product $\mathbf{D}(u)\mathbf{D}(-u)$ is diagonal, the entries depend on the boundary spins a_0, a_L. A graphical proof of these identities is given in Appendix A.

We emphasize that the inversion identities (2.10) and (2.14) are valid for any IRF model with bulk and boundary weights satisfying the algebraic relations (2.2)–(2.8).

3. Application to RSOS models

As an application of the identities derived above we now consider the critical restricted solid-on-solid (RSOS) models on a square lattice. The SOS Boltzmann weights satisfying the Yang–Baxter equation (2.2) are [34]

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = \delta_{bd}\sqrt{\frac{[a][c]}{[b][d]}}\rho(u+\lambda) - \delta_{ac}\rho(u), \qquad \rho(u) = \frac{\sin(u-\lambda)}{\sin\lambda}, \tag{3.1}$$

with $[x] = \sin(x\lambda)/\sin\lambda$, for height variables satisfying the SOS condition given in terms of the adjacency matrix A with entries $A_{ab} = \delta_{a,b+1} + \delta_{a,b-1}$. The Boltzmann weights satisfy the unitarity relation (2.3) and crossing symmetry[1]

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = \sqrt{\frac{[a][c]}{[b][d]}}\, W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| \lambda - u\right). \tag{3.2}$$

[1] Note that the crossing relation in the form (2.4) used above can be recovered by a gauge transformation of the Boltzmann weights [33]

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) \to \left(\frac{[a][c]}{[b][d]}\right)^{\frac{u}{2\lambda}} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right).$$

This gauge does not affect the transfer matrix (2.9) of the periodic model (nor, after a similar transformation of the boundary weights, that for open boundary conditions, Eq. (2.12)) or the corresponding inversion identities.

Choosing $\lambda = \pi/r$ and limiting the height variables to take values from $\mathfrak{S} = \{1, 2, \dots, r-1\}$ subject to the RSOS condition expressed through the resulting $(r-1) \times (r-1)$ adjacency matrix defines the restricted SOS model.

3.1. Periodic boundary conditions

Considering the RSOS model with inhomogeneities $\{u_k\}_{k=1}^L$ the transfer matrix is given by Eq. (2.9) and satisfies the $L-1$ independent inversion identities (2.10). The eigenvalues of $\mathbf{T}(u)$ satisfy a similar identity, i.e.

$$\Lambda^{(\mathrm{p})}(u_k)\Lambda^{(\mathrm{p})}(\lambda + u_k) = \prod_{\ell=1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k), \quad k = 1, \dots, L-1. \tag{3.3}$$

A similar set of equations has recently been obtained for the SOS model with antiperiodically twisted boundary conditions (corresponding to $a_L = r - a_0$ in the present context) model by extending Sklyanin's separation of variables method [21] to the corresponding dynamical six-vertex model [29]. In this approach the RHS of the inversion identity is related to the quantum determinant of the dynamical vertex model.

Using (2.10) or (3.3) together with some information on the analytical properties of the transfer matrix the solution of the spectral problem is possible: from (3.1) we find that the transfer matrix is periodic in u with period π (for even length lattices) and both $\mathbf{T}(u)$ and its eigenvalues can be written as Fourier polynomials

$$\mathbf{T}(u) = \sum_{n=-L/2}^{L/2} \mathbf{T}_n \mathrm{e}^{i2nu}. \tag{3.4}$$

The spectrum of the transfer matrix can be classified into $r-1$ topological sectors through the asymptotic behavior of the transfer matrix eigenvalues, i.e.

$$\left(\prod_{\ell=1}^{L} \mathrm{e}^{\pm i(u_\ell + \lambda/2)}\right) \Lambda^{(\mathrm{p})}_{\pm L/2} = \frac{\alpha^{(n)}}{(2\sin\lambda)^L}, \tag{3.5}$$

where $\alpha^{(n)}$ take values from the spectrum of the adjacency matrix, i.e. $\alpha^{(n)} \in \mathrm{spec}(A) = \{2\cos(a\lambda)\}_{a=1}^{r-1}$ [8]. Given a possible value of $\Lambda^{(n)}_{\pm L/2}$ the inversion identities (2.10) constitute a system of quadratic equations for the remaining Fourier coefficients. For small systems we have verified that they have $2^{(L-1)}$ independent solutions for each topological sector. This is more than the $\binom{L}{L/2}$ corresponding eigenvalues of the transfer matrix of the *unrestricted* solid-on-solid model. The RSOS spectrum is known to be the subset of the SOS one with eigenstates of non-zero norm on the restricted Hilbert space, i.e. $a \in \{1, 2, \dots, r-1\}$ [35]. We shall return to the identification of the RSOS spectrum among the solutions of the inversion identities below.

For an efficient computation of transfer matrix eigenvalues for large systems the identities (2.10), however, are not suitable. For a generic choice of the inhomogeneities, in particular $u_\ell \neq u_k + \lambda$ for $\ell \neq k$, however, they are formally equivalent to Baxter's TQ-equation

$$\Lambda^{(\mathrm{p})}(u)q(u) = a(u)q(u-\lambda) + d(u)q(u+\lambda), \tag{3.6}$$

restricted to the discrete set of points $u \in \{u_k, u_k + \lambda\}_{k=1}^L$ provided that $a(u_k) = 0 = d(\lambda + u_k)$ and, as a consequence of (3.3),

$$a(\lambda + u_k)d(u_k) = \prod_{\ell=1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k). \tag{3.7}$$

In the context of Sklyanin's separation of variables this amounts to a choice of $a(u)$, $d(u)$ factorizing the quantum determinant of a vertex model [21].

TQ-equations such as (3.6) holding for arbitrary u are obtained in the Bethe ansatz formulations of the spectral problem of integrable systems. Provided that they allow for a sufficiently simple (e.g. polynomial) ansatz for the functions $q(u)$ they can be solved using the Bethe equations for the finitely many zeroes of these functions.

In the present case of the periodic RSOS model we factorize (3.7) as

$$a(u) \equiv \omega \prod_{\ell=1}^{L} \frac{\sin(u - u_\ell)}{\sin\lambda}, \qquad d(u) \equiv \omega^{-1} \prod_{\ell=1}^{L} \frac{\sin(-u + u_\ell + \lambda)}{\sin\lambda}, \tag{3.8}$$

and take the Fourier polynomial

$$q(u) = \prod_{j=1}^{M} \sin(u - \mu_j), \tag{3.9}$$

parametrized by M complex numbers $\mu_j \equiv i\alpha_j + \lambda/2$ as our ansatz for the q-functions. As a consequence of the analyticity of the transfer matrix and its eigenvalues the α_j are determined by the Bethe equations of the inhomogeneous six-vertex model with twisted boundary conditions

$$\omega^2 \prod_{\ell=1}^{L} \frac{\sinh(\alpha_j + iu_\ell - \frac{i\lambda}{2})}{\sinh(\alpha_j + iu_\ell + \frac{i\lambda}{2})} = -\prod_{k=1}^{M} \frac{\sinh(\alpha_j - \alpha_k - i\lambda)}{\sinh(\alpha_j - \alpha_k + i\lambda)}, \quad j = 1, \dots, M. \tag{3.10}$$

Here the twist parameter ω has to be chosen such that the transfer matrix eigenvalues $\Lambda^{(p)}(u)$ obtained from the *TQ*-equation shows the asymptotic behavior (3.5), i.e.

$$\omega e^{i(\frac{L}{2}-M)\lambda} + \omega^{-1} e^{-i(\frac{L}{2}-M)\lambda} \equiv \cos a\lambda. \tag{3.11}$$

Finally, among the solutions to (3.10) the ones corresponding to eigenvalues of the RSOS model have to be selected. In previous studies of the RSOS model the set of Bethe equations (3.10) in the homogeneous limit $u_\ell \equiv 0$ has been obtained by embedding the RSOS transfer matrix into the fusion hierarchy of integrable generalizations of the RSOS model [6] and using the algebraic Bethe ansatz [28,35]. In Ref. [6] it has been conjectured, that the spectrum of the RSOS model is obtained from configurations of $M = L/2$ roots grouped into strings of length $n = 1, \dots, r-1$ with real centers $\alpha_j^{(n)}$ [36]

$$\alpha_{j,m}^{(n)} = \alpha_i^{(n)} + i\lambda\left(\frac{n+1}{2} - m\right), \quad m = 1, \dots, n. \tag{3.12}$$

Based on this conjecture, integral equations for the densities of these strings in the thermodynamic limit have been derived and the low energy effective field theories describing the critical behavior of the RSOS models have been identified [6,37].

3.2. Open diagonal boundary conditions

As a consequence of the adjacency condition for the RSOS model the most general boundary matrix has the form [33]

$$B\left(\begin{matrix} a \\ b \end{matrix} c \middle| u\right) = \delta_{a\neq b} X^c_{ab}(u) + \delta_{a,b}\big(\delta_{c,a+1} D_c(u) + \delta_{c,a-1} U_c(u)\big). \tag{3.13}$$

For diagonal boundaries (also named fixed boundaries [38]) the Hilbert space of the model can be decomposed into sectors labeled by the boundary heights a_0 and a_L, with the allowed combinations of (a_0, a_L) depending on the length L of the system. In this case the non-vanishing boundary weights are given by [7,33]

$$\begin{aligned} D_{a+1}(u) &= \sqrt{\frac{[a+1]}{[a]}} \frac{\sin(u+\xi_a)\sin(u-a\lambda-\xi_a)}{\sin^2\lambda}, \\ U_{a-1}(u) &= \sqrt{\frac{[a-1]}{[a]}} \frac{\sin(u-\xi_a)\sin(u+a\lambda+\xi_a)}{\sin^2\lambda}, \end{aligned} \tag{3.14}$$

for $2 \le a \le r-2$. The weights $D_2(u)$ and $U_{r-2}(u)$ simply multiply the eigenvalues of the transfer matrix in the sectors with $a_0 \in \{1, r-1\}$, and similar for a_L. Therefore, any crossing symmetric choice of these weights satisfies the reflection (2.6) and boundary crossing symmetry (2.8) relations.

The inversion identities of the transfer matrix are given by Eq. (2.14), with the boundary information being captured by the functions

$$\beta_a(u) = \frac{\sin(u-\xi_a)\sin(u+a\lambda+\xi_a)}{\sin^2\lambda}. \tag{3.15}$$

In general, the boundary parameters ξ_a are chosen different for the left and right boundaries. By construction, the double-row transfer matrix of the RSOS model (and its eigenvalues) is an even Fourier polynomials in u

$$\mathbf{D}(u) = \sum_{k=-L-2}^{L+2} \mathbf{D}_k \mathrm{e}^{2iku}, \quad \mathbf{D}_k = \mathbf{D}_{-k}\mathrm{e}^{-2ik\lambda}, \tag{3.16}$$

and becomes diagonal at the special point (note that $D^{a_0\cdots a_L}_{b_0\cdots b_L}(u) \propto \delta^{a_0}_{b_0}\delta^{a_L}_{b_L}$ for diagonal boundaries)

$$D^{a_0\cdots a_L}_{b_0\cdots b_L}(u=0) = 2\cos(\lambda)\beta_{a_0}(0)\beta_{a_L}(0)\left(\prod_{\ell=1}^{L}\rho(u_\ell)\rho(-u_\ell)\right). \tag{3.17}$$

The corresponding eigenvalues $\Lambda^{(\mathrm{o})}(u)$ will have a Fourier expansion similar to (3.16). Taking into account the crossing symmetry (2.13) inherited from the transfer matrix, $\Lambda^{(\mathrm{o})}(u) = \Lambda^{(\mathrm{o})}(\lambda - u)$, there are in total $L+3$ undetermined Fourier coefficients. The L inversion identities (2.14) and relation (3.17) may be supplemented by the asymptotic behavior of the transfer matrix: analyzing the spectra of the double-row transfer matrix for small lattice lengths L, we find that the leading term is

$$\Lambda^{(\mathrm{o})}_{L+2} = \frac{2\cos(\lambda)\mathrm{e}^{-i\lambda(L+2)}}{(2i\sin\lambda)^{2L+4}}, \tag{3.18}$$

independent of the boundary spins a_0, a_L or of the inhomogeneities of the lattice. The subleading Fourier coefficient $\Lambda^{(\mathrm{o})}_{L+1}$, however, does depend on the choice of the boundary sector (a_0, a_L) but is the same for all states within this sector. This fact implies that within a TQ-equation formulation, the subleading order should not depend on the Bethe roots. This observation together with the inversion identities (2.14) for the eigenvalues and Eqs. (3.17) and (3.18) provides a consistent set of relations, through which the $L+3$ Fourier coefficients are completely determined. We have used this scheme to determine the eigenvalues of the double-row transfer matrix of RSOS models with $r=4,5,6$ and systems sizes up to $L=8$. Comparison with the spectrum obtained by exact diagonalization of the transfer matrix exhibits perfect agreement.

Having verified, that the identities listed above do in fact capture the full information required for the computation of the transfer matrix spectrum, we now formulate a *TQ*-equation which allows for an efficient determination of the eigenvalues for arbitrary system sizes. Similar as in the case of periodic boundary conditions, Eq. (3.6), we note that the inversion identities (2.14) for the eigenvalues can be interpreted as conditions for the solvability of the difference equation

$$\Lambda^{(\mathrm{o})}(u)q(u) = a(\lambda-u)q(u-\lambda) + a(u)q(u+\lambda), \tag{3.19}$$

at the special values $u=u_k$ provided that $a(\lambda-u_k)=a(\lambda+u_k)=0$ and

$$\begin{aligned} a(u_k)a(-u_k) = {} & \beta_{a_0}(u_k)\beta_{a_0}(-u_k)\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\frac{\rho(-\lambda+2u_k)}{\rho(2u_k)}\frac{\rho(-\lambda-2u_k)}{\rho(-2u_k)} \\ & \times\prod_{j=1}^{L}\rho(u_k-u_j)\rho(-u_k-u_j)\rho(-u_k+u_j)\rho(u_k+u_j). \end{aligned} \tag{3.20}$$

In (3.19) we have used the crossing symmetry (2.13) of the transfer matrix and assumed that the q-function is given by the crossing symmetric Fourier polynomial

$$q(u) = \prod_{\ell=1}^{M}\sin(u-\mu_\ell)\sin(u+\mu_\ell-\lambda), \tag{3.21}$$

whose degree M is determined below.

To determine the functions $a(u)$ we require (3.20) to hold for generic values of the spectral parameter: together with the asymptotic behavior (3.18) of $\Lambda^{(\mathrm{o})}$ this condition is found to determine uniquely the bulk part (containing factors of $\rho(u)$) of $a(u)$. For the factorization of the boundary terms, containing the factors $\beta_\alpha(u)$, there exist four possible combinations, leading to different *TQ*-equations. It is convenient to parametrize these distinct combinations by introducing the signs $\{\epsilon_0,\epsilon_L\}\in\{\pm1\}$. The generic boundary dependence in $a(u)$ which encompasses all four possibilities can be written then as $\beta_{a_0}(\epsilon_0 u)\beta_{a_L}(\epsilon_L u)$ and the function $a(u)$ reads

$$a(u) = \beta_{a_0}(\epsilon_0 u)\beta_{a_L}(\epsilon_L u)\frac{\rho(-\lambda+2u)}{\rho(2u)}\prod_{j=1}^{L}\rho(u-u_j)\rho(u+u_j). \tag{3.22}$$

Comparison with the leading asymptotic behavior (3.18) relates uniquely the value of M with the boundary heights a_0, a_L for each combination

$$M = \frac{L-a_0\epsilon_0-a_L\epsilon_L}{2}. \tag{3.23}$$

We have numerically verified for small size systems that each of the resulting *TQ*-equations leads to the correct spectrum of the transfer matrix. Furthermore, the subleading order coefficient of the Fourier expansion of the eigenvalues (3.19) turns out to be independent of the signs ϵ_0,ϵ_L and of the Bethe roots μ_j

$$\Lambda_{L+1} = -\frac{2e^{i\lambda}\Lambda_{L+2}}{\cos\lambda}\biggl(\cos(a_0\lambda)\cos(a_0\lambda+2\xi_{a_0}) + \cos(a_L\lambda)\cos(a_L\lambda+2\xi_{a_L}) + \cos 2\lambda \sum_{\ell=1}^{L}\cos 2\mu_\ell\biggr), \tag{3.24}$$

as expected from our numerical analysis above. For the particular choice $\epsilon_0 = -\epsilon_L = -1$ the TQ-equation (3.19) coincides with the one derived for the SOS models via the algebraic Bethe ansatz starting from a reference state outside the Hilbert space of the RSOS model [33]. Using the analyticity of the transfer matrix one can derive Bethe equations for the parameters $\mu_j = i\alpha_j + \lambda/2$, $k = 1, \ldots, M$, in the q-function for the sector (a_0, a_L):

$$\begin{aligned}&\prod_{x=0,L}\frac{\sinh(\alpha_j - i(\epsilon_x\xi_{a_x} - \frac{\lambda}{2}))}{\sinh(\alpha_j + i(\epsilon_x\xi_{a_x} - \frac{\lambda}{2}))}\frac{\sinh(\alpha_j + i(\epsilon_x(\xi_{a_x} + a_x\lambda) + \frac{\lambda}{2}))}{\sinh(\alpha_j - i(\epsilon_x(\xi_{a_x} + a_x\lambda) + \frac{\lambda}{2}))}\\ &\quad\times\prod_{\ell=1}^{L}\frac{\sinh(\alpha_j - iu_\ell - \frac{i\lambda}{2})}{\sinh(\alpha_j - iu_\ell + \frac{i\lambda}{2})}\frac{\sinh(\alpha_j + iu_\ell - \frac{i\lambda}{2})}{\sinh(\alpha_j + iu_\ell + \frac{i\lambda}{2})}\\ &\quad=\prod_{k\neq j}^{M}\frac{\sinh(\alpha_j - \alpha_k - i\lambda)}{\sinh(\alpha_j - \alpha_k + i\lambda)}\frac{\sinh(\alpha_j + \alpha_k - i\lambda)}{\sinh(\alpha_j + \alpha_k + i\lambda)}.\end{aligned} \tag{3.25}$$

Here the first line contains the phase shifts associated with reflection from the left and right boundary, respectively.

3.3. Open non-diagonal boundaries

Boundary Boltzmann weights for RSOS models with non-diagonal (or free) integrable boundaries have been constructed by directly solving the BYBE [33,39] and by using the face–vertex correspondence [40]. In [38], an alternative realization of non-diagonal boundary weights was given based on an extension of the diagonal ones (3.14) with auxiliary face weights. As a consequence, the spectral problem of the RSOS models subject to non-diagonal boundary conditions can be mapped to that with diagonal ones presented in the previous section.

Following Ref. [38], the construction is based on the observation that if B satisfies the BYBE (2.6) then the dressed boundary weight B' defined by

$$\begin{aligned}&B'\left(\begin{matrix}a_0\\ b_0\end{matrix}\, c_0\middle|\, u\right)\\ &\quad= B\left(\begin{matrix}a_{-n}\\ b_{-n}\end{matrix}\, c_{-n}\middle|\, u\right)\\ &\qquad\times\sum_{c_{-1}\cdots c_{-n}}\left[\prod_{j=-n+1}^{0} W\left(\begin{matrix}a_{j-1} & a_j\\ c_{j-1} & c_j\end{matrix}\middle|\, u-u_j\right) W\left(\begin{matrix}c_{j-1} & c_j\\ b_{j-1} & b_j\end{matrix}\middle|\, \lambda-u-u_j\right)\right],\end{aligned} \tag{3.26}$$

also solves the BYBE. Here, the lattice has formally been extended by n faces with additional inhomogeneities $\{u_j\}_{j=-(n-1)}^{0}$. For a particular choice of the inhomogeneities the additional spin variables on sites $-n, \dots, -1$ can be eliminated using fusion projection operators on (3.26) [38].[2] Note, that as a consequence of the adjacency condition of the RSOS model all boundary spins a_0, b_0 have the same parity.

As an example, general non-diagonal left boundary weights for RSOS models with even parity boundary spins for $r > 4$ can be obtained by starting with the diagonal ones (3.14) for fixed boundary conditions $a_{-n} = b_{-n} \equiv \bar{a} = \frac{r}{2}$ $(\frac{r+1}{2})$ for even (odd) r, which are then dressed by $n = [\frac{r-1}{2}] - 1$ auxiliary faces. Projecting out the auxiliary spins requires to choose the auxiliary inhomogeneities in (3.26) as [38]

$$u_j = \chi_0 + (n + j - 1)\lambda, \quad j = -(n-1), \dots, 0. \tag{3.27}$$

Non-diagonal right boundary weights are constructed in an analogous way. The eigenvalues of the RSOS model with such boundary conditions can then be obtained from the *TQ*-equation Eq. (3.19) for an open RSOS model with $L + n_L + n_R$ faces. As a consequence of the constraint (3.27) the left boundary phase shift in (3.25) are changed to

$$\begin{aligned} &\frac{\sinh(\alpha - i(\epsilon_0 \xi_{a_0} - \frac{\lambda}{2}))}{\sinh(\alpha + i(\epsilon_0 \xi_{a_0} - \frac{\lambda}{2}))} \frac{\sinh(\alpha + i(\epsilon_0(\xi_{a_0} + a_0\lambda) + \frac{\lambda}{2}))}{\sinh(\alpha - i(\epsilon_0(\xi_{a_0} + a_0\lambda) + \frac{\lambda}{2}))} \\ &\quad \times \frac{\sinh(\alpha + i\chi_0 - \frac{i\lambda}{2})}{\sinh(\alpha - i\chi_0 + \frac{i\lambda}{2})} \frac{\sinh(\alpha + i\chi_0 - i(n_L - \frac{1}{2})\lambda)}{\sinh(\alpha - i\chi_0 + i(n_L - \frac{1}{2})\lambda)}, \end{aligned} \tag{3.28}$$

with a similar change for the right one.

4. Discussion

In this work we investigated inhomogeneous IRF models with different boundary conditions and were able to derive exact inversion identities satisfied by the commuting transfer matrices of these models. Since our derivation is based on a generic set of local relations satisfied by the face and boundary weights, these inversion identities are applicable for a large class of integrable IRF models. The identities found here are similar to the ones obtained previously for vertex models by means of separation of variables [16,18,21] or based on local properties of the vertex weights and reflection matrices [20].

Focusing then on the critical RSOS models with periodic and open boundary conditions, we have solved the spectral problem of the models by using the extracted inversion identities. In each case, the set of inversion identities once complemented with relations emerging from transfer matrix properties, such as their asymptotic behavior, forms a sufficient set to determine the eigenvalues of the latter. In a further step, the spectral problem has been formulated as a *TQ*-equation and a parametrization of the eigenvalues in terms of roots to Bethe equations can be derived.

For the periodic case, our results reproduce those obtained in Ref. [6] by means of functional relations arising from the fusion hierarchy of transfer matrices. For diagonal open boundary conditions we have derived four different *TQ*-equations, each of them yielding the complete spectrum of the RSOS transfer matrix. One of the *TQ*-equations found here coincides with that

2 A similar construction allows to construct dynamical (operator-valued) boundary matrices for vertex models by projection onto subspaces of the additional quantum spaces, see [41,42].

for the *unrestricted* SOS model with open diagonal boundary conditions by means of the algebraic Bethe ansatz [33]. We note, however, that this approach cannot be applied for the RSOS model due to the restriction of spin values. Furthermore, the embedding the transfer matrix of the latter into that of the SOS model has been shown for periodic boundary conditions only [35]. Finally, *TQ*-equations and a Bethe ansatz for RSOS models with general non-diagonal boundary conditions have been derived from the inversion identities for extended IRF models with diagonal boundary weights dressed by auxiliary faces [38].

These results for RSOS models show that the use of inversion identities provides a basis for the efficient solution of the spectral problem of IRF models in cases where other Bethe ansatz type approaches may not work. We note that our present analysis focused on the eigenvalues of the transfer matrix only. Eigenvectors (or the computation of matrix elements) have not been considered yet. Recently, methods related to the ones employed here have led to considerable advances in vertex models with non-diagonal boundary conditions [18,43,44]. Using face–vertex correspondence and, as a first step, exploring the formal similarity of our results to those obtained using separation of variables for dynamical vertex models [29] similar results can be expected for IRF models.

Another problem which has been successfully addressed using separation of variables for vertex models and the related spin-$\frac{1}{2}$ chains is that of the completeness of the Bethe ansatz [16, 21,27]. As has been discussed above, it easily seen that the number of solutions to the inversion identities exceeds the dimension of the Hilbert space for the RSOS models. This is a consequence of the restriction of spin variables and the constraints imposed by the adjacency condition. For the periodic RSOS model physical states have been associated with certain root patterns to the Bethe equations (3.10) [6]. For the RSOS model with open boundary conditions or more general IRF models further work is necessary for this classification.

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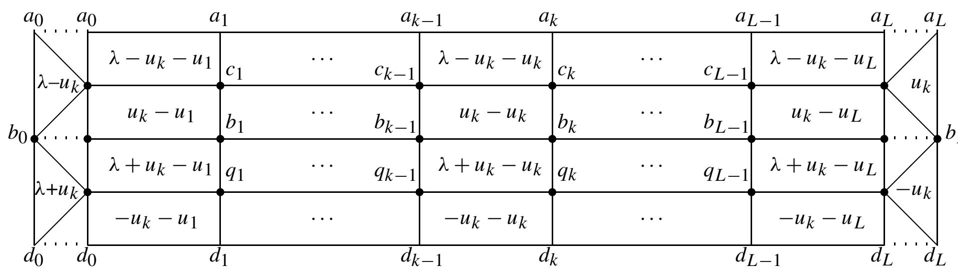
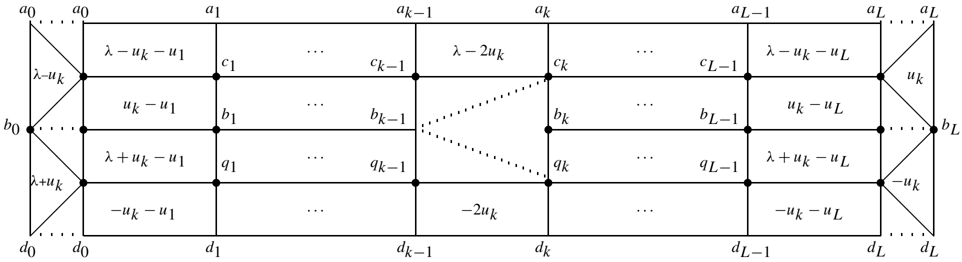
Appendix A. Graphical proof of the inversion identities (2.14) for open boundary conditions

For the double-row transfer matrix of the open boundary IRF model (2.12) we consider the product

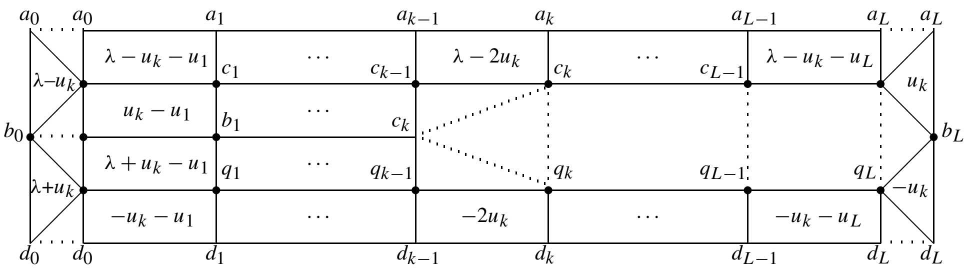
$$D^{a_0\cdots a_L}_{b_0\cdots b_L}(u_k)\,D^{b_0\cdots b_L}_{d_0\cdots d_L}(-u_k)$$

$=$ 

$=$ 

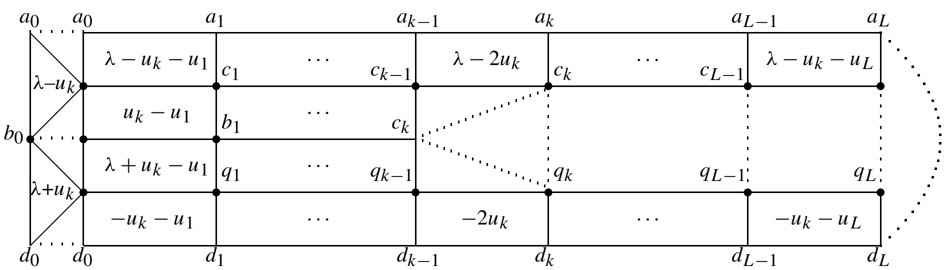
As in the periodic case summations over the spins $b_k, \dots, b_{L-1}$ can be performed using the unitarity condition (2.3) for the inner faces resulting in

$$\prod_{\ell=k+1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k)$$

$\times$ 

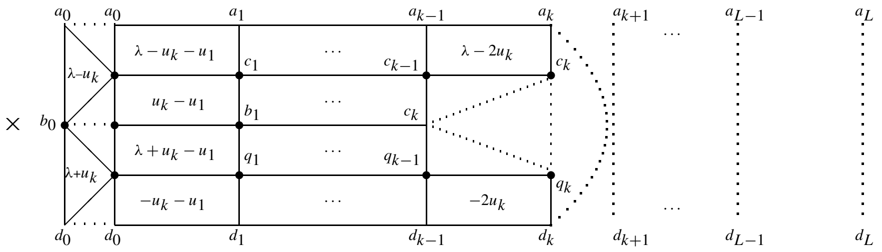
Next, summation over b_L with the boundary inversion condition (2.7) gives

$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k) \prod_{\ell=k+1}^{L} \rho(u_k - u_\ell)\rho(u_\ell - u_k)$$

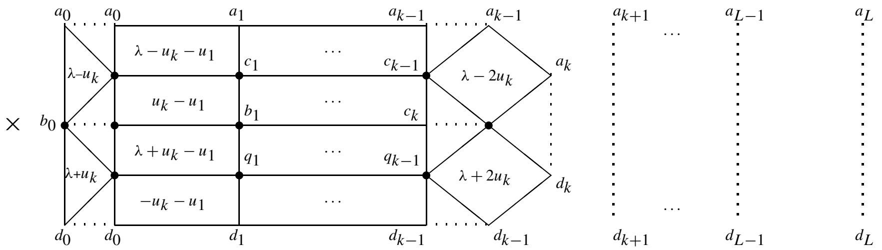
$\times$ 

Using unitarity again for the outer faces we perform the summation over q_ℓ, $\ell = L, L-1, \dots, k+1$

$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\prod_{\ell=k+1}^{L}\rho(u_k-u_\ell)\rho(-u_k+u_\ell)\rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$

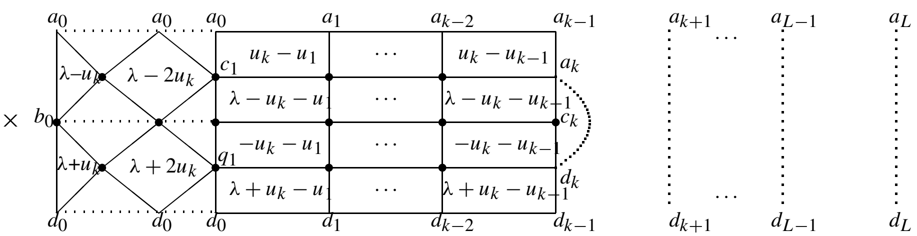


$$=\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\prod_{\ell=k+1}^{L}\rho(u_k-u_\ell)\rho(-u_k+u_\ell)\rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$



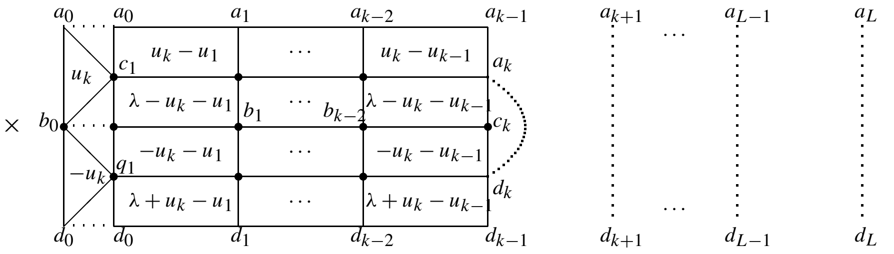
In the last step we have used the crossing symmetry (2.8) of the Boltzmann weights. Using the YBE (2.2) the outer and inner faces can be interchanged

$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\prod_{\ell=k+1}^{L}\rho(u_k-u_\ell)\rho(-u_k+u_\ell)\rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$



and using boundary crossing we obtain

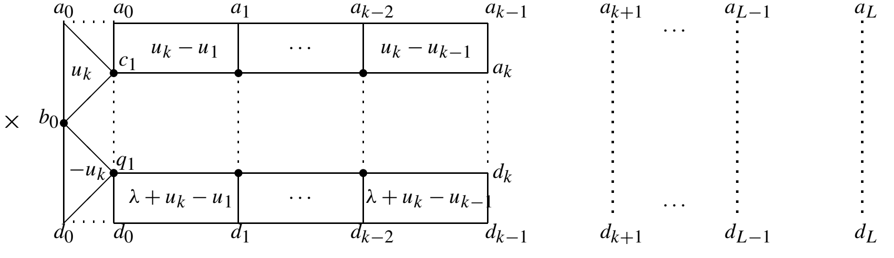
$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\rho(2u_k-\lambda)\rho(-\lambda-2u_k)$$
$$\times\prod_{\ell=k+1}^{L}\rho(u_k-u_\ell)\rho(-u_k+u_\ell)\rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$



Employing unitarity (2.3) for the inner faces the spins c_k and b_ℓ, $\ell = k-2, \dots, 1$ are summed over

$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\rho(2u_k-\lambda)\rho(-\lambda-2u_k)$$

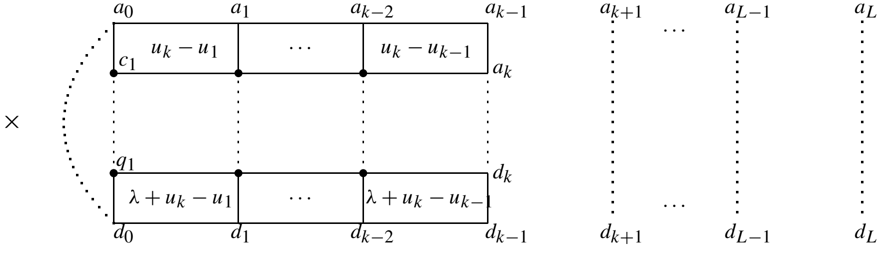
$$\times \prod_{\ell=k+1}^{L} \rho(u_k-u_\ell)\rho(-u_k+u_\ell) \prod_{\substack{\ell=1\\ \ell\neq k}}^{L} \rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$



Finally, using the boundary inversion condition (2.7) and unitarity of the outer faces the summations over the remaining internal spins can be done resulting in

$$\beta_{a_L}(u_k)\beta_{a_L}(-u_k)\beta_{a_0}(u_k)\beta_{a_0}(-u_k)\rho(2u_k-\lambda)\rho(-\lambda-2u_k)$$

$$\times \prod_{\ell=k+1}^{L} \rho(u_k-u_\ell)\rho(-u_k+u_\ell) \times \prod_{\substack{\ell=1\\ \ell\neq k}}^{L} \rho(u_k+u_\ell)\rho(-u_k-u_\ell)$$



$$= \beta_{a_L}(u_k)\beta_{a_L}(-u_k)\beta_{a_0}(u_k)\beta_{a_0}(-u_k)\rho(2u_k-\lambda)\rho(-\lambda-2u_k)$$
$$\times \prod_{\substack{\ell=1\\ \ell\neq k}}^{L} \rho(u_k-u_\ell)\rho(-u_k+u_\ell)\rho(u_k+u_\ell)\rho(-u_k-u_\ell) \times \delta_{d_0}^{a_0}\delta_{d_1}^{a_1}\cdots\delta_{d_L}^{a_L}.$$

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